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Grant A. Harter

Portland State University

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Implications of the Nambu Jona Lasinio Model with a New Regularization Renormalization
Method

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Grant Harter

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Thesis Adviser

Guang-Jiong Ni

Portland State University

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Implications of the Nambu Jona-Lasinio Model with a New Regularization Renormalization Method

Grant Harter¹ and

Advisor: Guang-Jiong Ni^{1,2}

¹*Department of Physics, Portland State University*

²*Department of Physics, Fudan University,*

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1 Abstract

In this paper, recent developments leading to a new formulation of the Nambu Jona-Lasinio (NJL) model for vacuum phase transitions (VPTs) which accounts for the divergence present in the original model, as well as generalizing it to multiple flavors of leptons, will be discussed. Originally the NJL model was conceived from a comparison between particle physics and the Bardeen Cooper Schrieffer (BCS) Theory of Superconductivity. Nambu and Jona-Lasinio used the component of Cooper pairs to model the behavior of a pair of massless fermions which condense in a vacuum via a VPT to form a massive fermion. However, Nambu and Jona-Lasinio found that their theory contained a divergent integral in the expression for the vacuum energy. They resolved this by introducing an ambiguous cutoff which prevented the acquisition of quantitative results. Using a new regularization renormalization method proposed by Ni's colleague and former student, Yang, Ni disposed of this cutoff and allowed for the NJL model to produce quantitative results. However, discrepancies in the final vacuum

energy density, as well as the lack of generalization from this theory to multiple flavors of leptons indicated that more research was necessary to propose a complete NJL theory. New research conducted by Ni and his collaborators (including this author) generalizes the NJL model by taking the vacuum energy as a sum over all of the multiple flavors of leptons. This same approach of using the vacuum energy has been taken to applying the NJL model to quarks. Computational solutions to the mass ratio equations proposed by the NJL model have not yet been obtained. These computational results shall be used to confirm the validity of the proposed theoretical framework.

2 Introduction

In 1961, Yoichiro Nambu and Giovanni Jona-Lasinio proposed the NJL model for vacuum phase transitions[[1],[2]]. It described the process through which a pair of massless fermions (particles which obey Fermi-Dirac statistics, such as quarks and leptons) undergo a VPT to become a massive particle. This theory was based in part on an analogy to the BCS theory of superconductivity proposed by John Bardeen, Leon Cooper, and Robert Schrieffer in 1954.

Although the qualitative value of the model was quickly seen by scientists, a quantitative appreciation was unobtainable. The main cause of this difficulty was the presence of the quadratically divergent integral I (Eq 2.1)

$$I = \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + \Delta^2}} \quad (2.1)$$

in one of the main mathematical components of the NJL model, the gap equation (Eq 2.2).

$$1 = \frac{4G}{V} \sum_{\mathbf{p}} \frac{1}{E_p} = \frac{2G}{\pi^2} I \quad (2.2)$$

Because the gap equation is responsible for describing the process by which a particle's mass condenses as the VPT proceeds, the divergent integral is particularly damaging to the theory. In the original theory, Nambu and Jona-Lasinio proposed that the integral could be solved by using a cutoff constant Λ . In other words, rewriting the integral as Eq 2.3.

$$I = \int_0^\Lambda dp \frac{p^2}{\sqrt{p^2 + \Delta^2}} \quad (2.3)$$

However, because the cutoff constant is ambiguous, the solution to this equation, and indeed the final results of the theory, are ambiguous as well, obstructing the theory from use quantitatively.

Due to this complication, for over fifty years subsequent to its proposal, the NJL model remained of little use quantitatively. However, in 1995, Ji Feng Yang, a PhD student of Guang-Jiong Ni (a professor at Fudan University), presented in his PhD thesis [[3]] a new regularization renormalization method for divergent integrals. This method proposed creating a differential equation by taking the derivative of the integral until it is no longer divergent. Then, the differential equation would be solved to give the solution to the integral. In 2013, after its successful application to such topics as the calculation of the mass of the Higgs particle [[4], [5]], Ni, Yang, Jianjun Xu, and Senyue Lou applied the method to the divergent integral in the gap equation of the NJL mode [[6]]

Despite the success of this endeavor, problems still remained with the new form of the NJL model. First, it only described the transition of one flavor of lepton, the electron. In order to fully describe the VPTs of leptons, the model must at least describe the additional leptons: the muon and the tau. The second problem with this new version of the NJL model was that it contained a decrease in the energy density of the vacuum within which the VPT occurred, after its completion. It was proposed by Ni that this energy decrease may be rectified by the consideration of the muon and the tau. Currently, this author is working with Ni and Ni's collaborators to develop this new theory.

The purpose of this paper is to review the background of the NJL model, and examine some of the research currently being done to develop a complete theory of how leptons undergo VPTs. Section 3 will give a brief introduction to Quantum Field Theory and its relevancy to the NJL model. Section 4 will give a short summary of the BCS theory of superconductivity and its relevancy to the NJL model. Section 5 will give a short summary of the portions of the NJL model as originally proposed by Nambu and Jona Lasinio, which are relevant to this research. Section 6 will outline Ni's 2013 contribution to the NJL model. Section 7 will explore the research currently being conducted by Ni and his colleagues (including this author).

3 Quantum Field Theory

In order to properly examine the NJL model, it is important to first become familiar with Quantum Field Theory (QFT). First formulated in the early nineteen hundreds by numerous physicists (notably Einstein, Dirac, Pauli, Fermi, Feynman, Dyson, Tomonaga, Schwinger, etc.), QFT concerns multiple particles created from a vacuum and traveling at relativistic speeds. The fundamental concept behind QFT is that particles are excited states of a physical field. One kind of particles is described by one kind of field in QFT.

The first is the Fermi-Dirac Field. Particles existing as a part of this field are called Fermions and obey Fermi-Dirac statistics (a set of physical rules which the particles follow). Numerous composite particles are fermions, however the main elementary particles which are considered Fermions are the quarks and the leptons. The key property of Fermi-Dirac statistics is that no two particles can exist in the same quantum state. A way to think of this is by an analogy with classical physics. In classical physics, no two objects can be at the same place at the same time. This notion could be understood to evolve into the Pauli exclusion principle in Quantum Mechanics and in a way, this is the requirement of Fermi-Dirac statistics [[7]].

The second field is the Bose-Einstein field. In the Bose-Einstein field, particles obey a different set of rules (Bose-Einstein statistics). Unlike the Fermi-Dirac field, Bose-Einstein statistics do not impose the condition that two particles in the field cannot exist in the same quantum state. Particles which make up this field are called bosons. As with Fermions, Bosons consist of numerous composite particles. However, the only elementary particles which are bosons are the photon, the gluon, the weak force bosons and the (hypothetical) graviton. Recently, the Higgs boson's theoretical prediction was confirmed by the Large Hadron Collider, leading to its inclusion as the fifth noncomposite boson in the standard model (not including the graviton due to its hypothetical nature) [[7]].

The NJL model for VPTs as it was originally proposed applied only to the electron. Ni's 2013 application of the regularization renormalization method, although addressing the lack of generalization, did not attempt to propose a solution generalized to all fermions. However, the current research being conducted by Ni and his colleagues (including this

author) is attempting to address this issue. Beginning with leptons, the research group is attempting to generalize the NJL model to all of the elementary fermions and the 0^+ scalar bosons (not included in this paper).

4 The Theory of Superconductivity

In 1954 John Bardeen, Leon Cooper, and Robert Schrieffer proposed the BCS theory of superconductivity [[8], [9]]. The development of this theory proved to be a groundbreaking achievement following the long struggle to find a suitable theory for the phenomena. Earlier that year, Leon Cooper set the framework for the theory by publishing a paper on what are now known as Cooper pairs [[10]]. The BCS theory was integral to the development of the NJL Model, as it was by analogy to the Cooper pairs that the NJL model was originally conceived.

The BCS theory of superconductivity describes the process by which a material becomes a superconductor. A superconductor is defined as a material with virtually zero electrical resistance. Certain materials (primarily metals) will become superconductors below a certain temperature (which is different per material).

Taking a step back for a moment, when an electron travels through an atomic lattice, it behaves similarly to a photon traveling out of a star, it bounces about. Instead of taking a direct path through the lattice, impurities and phonons (vibrations in the lattice structure) cause the electron to deviate. Because current is defined as the flow of electrons through the material, if the electrons are being prevented from quickly passing through the lattice, the current is obstructed and needs to be pushed by an external electric field. The cause of this current lessening, the composition of the phonons and lattice impurities, is resistance.

On the other hand, current is carried through a superconductor by Cooper pairs. Cooper pairs are electrons in a bound state despite their opposite momentum directions and spin orientations. The nature of this odd coupling of attractive force comes from the following reason. When an electron interacts with a phonon (a vibrating nucleus in the lattice), it pulls the positive protons in the nucleus toward it. This means that an electron traveling toward the nucleus from the other direction is caught in the potential well left by the positive

nucleus' movement. This causes the coupling of the caught electron and the electron which is attracted to the nucleus. This pair of electrons is a Cooper pair.

Below a certain temperature, all resistance in the electron lattice disappears. This is because, according to the BCS theory, the wavefunctions of multiple traveling Cooper pairs become coupled. So, when a Cooper pair passes a discontinuity in the lattice, though it wants to scatter, the amount of energy it would take to do so would be unachievable (it would, in fact, require the destruction of not only the pair, but many other pairs around it). So, all of the Cooper pairs enter and exit the lattice without any scattering or phonon interactions, leading to an uninhibited current and thus zero resistance.

It was in examining the theory of Cooper Pairs that Nambu and Jona-Lasinio were able to create their NJL theory. They compared the coupling of the Cooper pairs with the coupling between two massless fermions undergoing a VPT. Under this analogy, they devised a model for the behavior of these massless fermions.

5 The Nambu Jona Lasinio Model

This model (the NJL model), as has already been stated, describes the process by which massless lepton-antilepton pairs attract each other so that the vacuum undergoes a VPT, rendering the creation of a massive lepton possible [[1], [2]]. The mass, m , of the final lepton is given in the NJL model in terms of the g_0 and Λ . Λ is a cutoff introduced to curb the divergence eventually present in the theory. g_0 is a coupling constant with a dimension of $[M]^{-2}$, called the bare coupling constant and is present in the Lagrangian density (\mathcal{L} with a dimension of $[M]^4$) of the system (Eq 5.1).

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + g_0[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \quad (5.1)$$

Where ψ has a dimension of $[M]^{\frac{3}{2}}$. γ_5 in Eq 5.1 represents the chirality operator (chirality represents the polarization direction of the particle in a Dirac field). Nambu and Jona-Lasinio said the Eq 5.2

$$m - m_0 = \Sigma(p, m, g, \Lambda) |_{i\gamma \bullet p + m = 0} \quad (5.2)$$

should be satisfied for any Fermion which is real, where Σ represents the unrenormalized proper self energy associated with the fermion. They used the relationship $g/g_0 = \Gamma(m, g, \Lambda)$ between the bare coupling constant and the coupling constant in Eq 5.2. This relationship gives that the fraction is a function of the mass, the coupling constant, and the cutoff used.

From this, an expression for m was found to be

$$m = -\frac{g_0 m i}{2\pi^4} \int \frac{d^4 p}{p^2 + m^2 - i\epsilon} F(p, \Lambda) \quad (5.3)$$

with $F(p, \Lambda)$ being a factor of the cutoff. The only nontrivial solution to Eq 5.3 is

$$1 = -\frac{g_0 i}{2\pi^4} \int \frac{d^4 p}{p^2 + m^2 - i\epsilon} F(p, \Lambda). \quad (5.4)$$

A solution was found after Nambu and Jona-Lasinio set $p^2 = \Gamma^2$. The solution they proposed was:

$$\frac{2\pi^2}{g_0 \Lambda^2} = 1 - \frac{m^2}{\Lambda^2} \ln \left(\frac{\Lambda^2}{m^2} + 1 \right) \quad (5.5)$$

In order for this to have a real solution for $\frac{\Lambda}{m}$, the condition

$$0 < \frac{2\pi^2}{g_0 \Lambda^2} < 1 \quad (5.6)$$

must be true.

Unfortunately, as Nambu and Jona-Lasinio stated themselves, this only gives an approximation. The actual results of the NJL model cannot be quantitatively viewed using a cutoff constant. This problem would haunt physicists for many years, until the use of Yang's regularization renormalization method.

6 Generalized Nambu Jona Lasinio Model

Ni was unsettled by the ambiguous cutoff in Eq 5.6. Its presence prohibited any definite qualitative results from being gathered from the NJL model. In 2013, Ni and his colleagues proposed the implementation of a new method for the replacement of this cutoff.

6.1 Regularization Renormalization Method

The method used by Ni and his colleagues was first proposed by Ni's PhD student, Ji-Feng Yang, in his 1993 PhD dissertation [[3]], and subsequently used in a number of calculations including work related to the Higgs Particle. The fundamental concept behind Yang's regularization renormalization method is the differentiation (sometimes multiple times for integrals of quadratic and higher order divergence) of a divergent integral with respect to a parameter separate from the parameter being integrated by. The resulting differential equation is then solved, with the solution in terms of unknown arbitrary constants. These unknown constants are then solved for using the physical conditions of the system.

A simple example of this regularization renormalization method is provided by Ni, Xu, and Lou in their 2010 unpublished paper [[11]]. An integral used for Feynman diagrams is given as:

$$-i\Sigma(p) = (ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{ik^2} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu \quad (6.1)$$

Equation 6.1 can be transformed into:

$$-i\Sigma(p) = -e^2 \int_0^1 dx [-2(1-x)\not{p} - m] I \quad (6.2)$$

where:

$$I = \int \frac{d^4K}{(2\pi)^4} \frac{1}{(K^2 - M^2)^2} \quad (6.3)$$

with:

$$M^2 = p^2 x^2 + (m^2 - p^2)x \quad (6.4)$$

using $k \rightarrow K = k - xp$.

The integral I is divergent. Yang's regularization renormalization method gives that, I can be differentiated with respect to M^2 . The integral can then be conducted and the differential equation solved by integration to find:

$$I = \frac{-i}{(4\pi)^2} \left(\ln \frac{M^2}{\mu_2^2} + C_1 \right) \quad (6.5)$$

Which is in terms of the unknown constants μ_2 and C_1 . These two constants can then be combined into the constant μ_2 . Substituting Eq 6.5 into Eq 6.2, 6.2 can now be integrated. Using the physical conditions of the system, μ_2 can be found to be:

$$\mu_2 = me^{-5/6} \quad (6.6)$$

for a freely moving electron with observable mass m .

6.2 The Modified NJL model

Utilizing this same method for the NJL model, Ni began by giving a reformulated version of the gap equation [[6]]:

$$1 = \frac{4G}{V} \sum_{\mathbf{p}} \frac{1}{E_p} = \frac{2G}{\pi^2} I \quad (6.7)$$

using:

$$I = \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + \Delta^2}} \quad (6.8)$$

The first derivative integral of I with respect to Δ^2 is still divergent, but the second derivative is convergent, and the integral of the differential equation with respect to Δ^2 gives:

$$I = \frac{1}{4} \left[\Delta^2 \left(\ln \frac{\Delta^2}{\mu_s^2} - 1 \right) + C_2 \right] \quad (6.9)$$

in terms of C_2 and μ_s^2 . This can be substituted into Eq. 6.7 to give:

$$\left[\Delta^2 \left(\ln \frac{\Delta^2}{\mu_s^2} - 1 \right) + C_2 \right]_{VPT} = \frac{2\pi^2}{G}, \quad C_2 = \frac{2\pi^2}{G} - \Delta^2 \left(\ln \frac{\Delta^2}{\mu_s^2} - 1 \right) \quad (6.10)$$

for the case when the VPT is finished.

Next, Ni proposed that the RRM could also be applied to the expression for the energy of the vacuum¹:

$$E_{vac} = 4 \sum_{\mathbf{p}} \omega_p V_p^2 - \frac{16G}{V} \left[\sum_{\mathbf{p}, \mathbf{p}'} (U_p V_p)(U_{p'} V_{p'}) \right] + \frac{8G}{V} \sum_{\mathbf{p}} U_p^2 V_p^2 \quad (6.11)$$

¹The operators U and V relate to the creation and destruction operators of the particle and the antiparticle and will be touched upon in Section Seven

where ω_p is the magnitude of the linear momentum described by:

$$\omega_p = ||\vec{\mathbf{p}}|| \quad (6.12)$$

The kinetic energy term of the energy expression is:

$$4 \sum_{\mathbf{p}} |\mathbf{p}| n(\mathbf{p}) = \frac{V}{\pi^2} J + \frac{\Delta^2 V}{\pi^2} I \quad (6.13)$$

and the integral J :

$$J = \int_0^\infty dp (p^3 - p^2 \sqrt{p^2 + \Delta^2}) \quad (6.14)$$

is quartically divergent. Again, using Yang's RRM and differentiating three times with respect to (wrt) Δ^2 , the result becomes:

$$J = -\frac{1}{16} \left[\Delta^4 \left(\ln \frac{\Delta^2}{\mu_s^2} - \frac{3}{2} \right) + 2C_2' \Delta^2 + C_3 \right] \quad (6.15)$$

where μ_s^2 is the same as from I , as it is a mass scale. The condition:

$$E_{vac}(\Delta^2 \rightarrow 0) = \langle 0 | \hat{H} | 0 \rangle = 0 \quad (6.16)$$

can be imposed to give the regularization condition:

$$C_3 = 0. \quad (6.17)$$

The vacuum state energy is then given in terms of Δ , μ_s , C_2 , and C_2' to be:

$$E_{vac} = -\frac{V}{16\pi^2} \left\{ \Delta^4 \left(\ln \frac{\Delta^2}{\mu_s^2} - \frac{3}{2} \right) + 2C_2' \Delta^2 - 4\Delta^2 \left[\Delta^2 \left(\ln \frac{\Delta^2}{\mu_s^2} - 1 \right) + C_2 \right] + \frac{G\Delta^2}{\pi^2} \left[\Delta^2 \left(\ln \frac{\Delta^2}{\mu_s^2} - 1 \right) + C_2 \right]^2 \right\} \quad (6.18)$$

Four conditions can be imposed to give the remaining unknown constants. The first is in evaluating the rate of change of the vacuum energy wrt Δ^2 when the VPT is finished ($\Delta \rightarrow \Delta_1$). At that point, the rate of change would be zero, implying that the vacuum state is stable.

$$\frac{\partial E_{vac}}{\partial \Delta^2} \Big|_{VPT} = 0 \quad (6.19)$$

This condition leads to the revelation that $C_2^2 = C_2$. Next, the condition that the vacuum energy is less than zero after the VPT (this is true because the system would lose energy to the vacuum phase transition) is imposed to give:

$$\ln \frac{\Delta_1^2}{\mu_1^2} < \frac{1}{2} \quad (6.20)$$

The next stable condition for the vacuum, to be imposed is:

$$\frac{\partial^2 E_{vac}}{\partial(\Delta^2)^2} \big|_{VPT} = 0 \quad (6.21)$$

This condition leads to the equation:

$$\Delta_1^2 = \mu_s^2 e^{-\frac{\pi^2}{G\Delta_1^2}} \quad (6.22)$$

The second to last condition applied to the system is that:

$$\frac{\partial^3 E_{vac}}{\partial(\Delta^2)^3} \big|_{VPT} = 0 \quad (6.23)$$

This condition leads to the fact that, in the case when Eq 6.23 is equal to zero:

$$\ln \frac{\Delta_1^2}{\mu_s^2} = -\frac{2}{3} \quad (6.24)$$

One final condition is also considered. This condition says that:

$$\frac{\partial^4 E_{VAC}}{\partial(\Delta^2)^4} > 0 \quad (6.25)$$

Based on these conditions, the final result of the revised NJL model gives that after the VPT, the relation between Δ_1 , μ_s , and the coupling constant G becomes:

$$G = \frac{3\pi^2}{2\Delta_1^2} = \frac{3\pi^2}{2\mu_s^2} e^{\frac{2}{3}} \quad (6.26)$$

Hence Δ^2 , the condensate, serves as a, "running order parameter," describing the VPT. Its final stable value shows up as the mass created. Unfortunately, despite the success of the regularization renormalization method, the NJL model still contained difficulties. The first is that it did not account for the muon and the tau. The model only addresses a VPT producing one flavor of lepton (the electron). The second was that Ni realized the figures contained within the final paper were flawed in two ways. The first was that gamma rays were predicted to be released by the VPT, while that would not be possible. The second difficulty was that the final vacuum state should not contain any massless particles, while in the expanded NJL theory proposed by Ni, it does. This led to the proposal of the seed mass (as examined in the next section).

7 Current Developments in the Nambu Jona Lasinio Model

In order to account for these difficulties, Ni and his colleagues (including this author) have begun to propose a new generalized NJL model. This author's contribution to this model was in performing relevant calculations and derivations alongside Ni's collaborators. This new model differs from the previous iteration of the NJL model, as the energy is a summation with respect to the number of flavors of particles. By accounting for these multiple flavors, this summation takes into account the energy density discrepancies present in the prior revision to the NJL model.

The generalized NJL model proposed by Ni and his research group relies on the Hamiltonian. The annihilation and creation operators respectively for the lepton given as \hat{a} and \hat{a}^\dagger are used, while the annihilation and creation operators respectively are taken to be \hat{b} and \hat{b}^\dagger , for the antilepton. These operators describe the creation and annihilation process of each flavor of fermion, which is why the use of the Hamiltonian rather than the Lagrangian is so important to the NJL model and its generalization. The construction of the Hamiltonian (\hat{H}) proves to be the first step crucial to the theory. The Hamiltonian is proposed to be crucial to the theory:

$$\hat{H} = \hat{H}_1 + \hat{H}_0 \quad (7.1)$$

\hat{H}_0 can be generalized to be:

$$\hat{H}_0 = \sum_{\mathbf{p}, h, l} \epsilon_{lp} \left(\hat{a}_{\mathbf{p}hl}^\dagger \hat{a}_{\mathbf{p}hl} + \hat{b}_{\mathbf{p}hl}^\dagger \hat{b}_{\mathbf{p}hl} \right) \quad (7.2)$$

ϵ_{lp} is $\epsilon_{lp} = \sqrt{\delta_l^2 + \omega_p^2}$, where δ_l is the seed mass of each lepton (the subscript of l defines the lepton considered), and ω_p is the magnitude of momentum, given by:

$$(\omega_p = |\mathbf{p}|) . \quad (7.3)$$

The anticommutation relations for fermions by definition read:

$$\{a, a^\dagger\} \equiv \left[\hat{a}_{\mathbf{p}hl}, \hat{a}_{\mathbf{p}'h'l'}^\dagger \right]_+ \equiv \left[\hat{a}_{\mathbf{p}hl} \hat{a}_{\mathbf{p}'h'l'}^\dagger + \hat{a}_{\mathbf{p}'h'l'}^\dagger \hat{a}_{\mathbf{p}hl} \right] = \delta_{ll'} \delta_{\mathbf{p}\mathbf{p}'} \delta_{hh'} \quad (7.4)$$

\hat{H}_1 would then be generalized as:

$$\hat{H}_1 = -\frac{1}{V} \sum_{\mathbf{p}, \mathbf{p}', h, h', l, l'} g_l g_{l'} h h' : \left[\left(\hat{a}_{\mathbf{p}hl}^\dagger \hat{b}_{-\mathbf{p}hl}^\dagger + \hat{b}_{-\mathbf{p}hl} \hat{a}_{\mathbf{p}hl} \right) \left(\hat{a}_{\mathbf{p}'h'l'}^\dagger \hat{b}_{-\mathbf{p}'h'l'}^\dagger + \hat{b}_{-\mathbf{p}'h'l'} \hat{a}_{\mathbf{p}'h'l'} \right) \right] : \quad (7.5)$$

where h is the helicity and is defined by:

$$h = \frac{\vec{\sigma} \cdot \vec{\mathbf{p}}}{|\vec{\mathbf{p}}|} \quad (7.6)$$

and can be either 1 or -1 . The coupling constant g_l is:

$$g_l^2 = G_l \sim [M]^{-2}, g_l \sim [M]^{-1} \quad (7.7)$$

The NJL vacuum state is the vacuum present after the VPT, and is generalized as:

$$|NJLV\rangle \equiv |0_{NJL}\rangle = \prod_{\mathbf{p}, h, l} \left(U_{pl} + h V_{pl} \hat{a}_{\mathbf{p}hl}^\dagger \hat{b}_{-\mathbf{p}hl}^\dagger \right) |0\rangle \quad (7.8)$$

From the NJL vacuum state, the NJL transformations can be generalized as:

$$\begin{aligned} \hat{\alpha}_{\mathbf{p}hl}^\dagger &= U_{pl} \hat{a}_{\mathbf{p}hl}^\dagger - h V_{pl} \hat{b}_{-\mathbf{p}hl} & \hat{a}_{\mathbf{p}hl}^\dagger &= U_{pl} \hat{\alpha}_{\mathbf{p}hl}^\dagger + h V_{pl} \hat{\beta}_{-\mathbf{p}hl} \\ \hat{\beta}_{-\mathbf{p}hl} &= U_{pl} \hat{b}_{-\mathbf{p}hl} + h V_{pl} \hat{a}_{\mathbf{p}hl}^\dagger & \hat{b}_{-\mathbf{p}hl} &= U_{pl} \hat{\beta}_{-\mathbf{p}hl} - h V_{pl} \hat{\alpha}_{\mathbf{p}hl}^\dagger \end{aligned} \quad (7.9)$$

The variables U_{lp} and V_{lp} are normalized as:

$$U_{lp}^2 + V_{lp}^2 = 1 \quad (7.10)$$

The NJL transformations can be substituted into the formula for the Hamiltonian (given in Eq 7.1, Eq 7.2, and Eq 7.5), which becomes:

$$\hat{H} = E_{vac} + \hat{H}_1 + \hat{H}_2 + \hat{H}_4 \quad (7.11)$$

The vacuum state energy E_{vac} is defined as the Hamiltonian operating on the NJL vacuum state as given by:

$$E_{vac} = \langle 0_{NJL} | \hat{H} | 0_{NJL} \rangle \quad (7.12)$$

The component \hat{H}_1 of the new Hamiltonian given in terms of the NJL transforms is:

$$\hat{H}_1 = \sum_{\mathbf{p}, h, l} E_{pl} \left(\hat{\alpha}_{\mathbf{p}hl}^\dagger \hat{\alpha}_{\mathbf{p}hl} + \hat{\beta}_{\mathbf{p}hl}^\dagger \hat{\beta}_{\mathbf{p}hl} \right) \quad (7.13)$$

with the energy of a massive particle after VPT being E_{pl} :

$$E_{pl} = \sqrt{p^2 + \tilde{m}_l^2} \quad (7.14)$$

where \tilde{m}_l^2 is given by Eq 7.20. The component \hat{H}_2 of the new Hamiltonian can also be written in terms of the NJL transformations as:

$$\hat{H}_2 = \sum_{\mathbf{p}, h, l} \left\{ 2\epsilon_{pl} U_{pl} V_{pl} - \frac{8g_l}{V} \left(\sum_{\mathbf{p}', l'} g_{l'} U_{p'l'} V_{p'l'} \right) (U_{pl}^2 - V_{pl}^2) + \frac{4G_l}{V} U_{pl} V_{pl} (U_{pl}^2 - V_{pl}^2) \right\} \left(h\hat{\alpha}_{\mathbf{p}hl}^\dagger \hat{\beta}_{-\mathbf{p}h\bar{l}}^\dagger + h\hat{\beta}_{-\mathbf{p}h\bar{l}} \hat{\alpha}_{\mathbf{p}hl} \right) \quad (7.15)$$

The inner summation term in \hat{H}_2 gives the gap equation:

$$\Delta_l = \frac{8g_l}{V} \left(\sum_{\mathbf{p}', l'} g_{l'} U_{p'l'} V_{p'l'} \right). \quad (7.16)$$

In order to construct a useful generalized version of the revised NJL model for VPTs, it is necessary to define a few useful constants. The ratios of the final masses squared of the leptons after the VPT has finished are important values, which will be denoted by r :

$$\frac{m_2}{m_1} = \frac{\Delta_2}{\Delta_1} = \frac{g_2}{g_1} = \sqrt{r} \quad (7.17)$$

and R :

$$\frac{m_3}{m_1} = \frac{\Delta_3}{\Delta_1} = \frac{g_3}{g_1} = \sqrt{R}. \quad (7.18)$$

It is important to note for these ratios that one of the lepton masses, m_1 , must be chosen as a datum (reference mass) as compared to the other leptons. On the same note, the mass ratio x is denoted as the ratio between the running mass and the reference mass:

$$x = \frac{\tilde{m}_1^2}{m_1^2} \quad (7.19)$$

The running mass \tilde{m}_1 is the mass of the reference mass lepton as it is created through the VPT:

$$\tilde{m}_l^2 = \Delta_l^2 + \delta_l^2 \quad (7.20)$$

The running mass approaches m_1 as the VPT progresses, meaning that x goes to 1 as the VPT goes to completion. The variable s on the other hand, is the ratio of the seed mass, δ_1 to the datum mass:

$$s = \frac{\delta_1^2}{m_1^2} \quad (7.21)$$

Similar to, but different from, the running mass, \tilde{m}_1^2 , the seed mass, δ_1 , denotes the mass which is going to the final mass, m_1 , but at the expense of decreasing Δ_1 according to Eq 7.20. This means that the VPT has s going from 0 to 1, while keeping $\Delta_l^2 \geq 0$. Various combinations of the r , R , s , and x also prove useful and are given below:

$$\frac{\Delta_l^2}{m_1^2} = x - s > 0 \quad (7.22)$$

$$\frac{\tilde{m}_2^2}{m_1^2} = \frac{\Delta_2^2 + \delta_2^2}{m_1^2} \quad (7.23)$$

$$\frac{\Delta_2^2}{m_1^2} = r(x - s) \quad (7.24)$$

$$\frac{\Delta_3^2}{m_1^2} = R(x - s) \quad (7.25)$$

It is important to note that for vacuum stability, the component of the Hamiltonian, \hat{H}_2 must be equal to zero. This condition is imposed because, should \hat{H}_2 not be equal to zero, it would indicate that particle antiparticle pairs would be automatically created and destroyed in the vacuum. This could not be true in a stable NJL vacuum state.

The variables U_{lp} and V_{lp} can be parameterized as a function of the variable θ_{lp} according to Eq 7.10. The parameterizations are given as:

$$U_{lp} = \cos\theta_{lp} \rightarrow 1 \quad (7.26)$$

$$V_{lp} = \sin\theta_{lp} \rightarrow 0 \quad (7.27)$$

The parameterization also implies that the tangent of $2\theta_{lp}$ is equal to:

$$\tan 2\theta_{lp} = \frac{\Delta_l}{\varepsilon_{lp}} \quad (7.28)$$

and that the sine of $2\theta_{lp}$ is equal to:

$$\sin 2\theta_{lp} = 2U_{lp}V_{lp} = \frac{\Delta_l}{E_{lp}} \rightarrow 0 \quad (7.29)$$

and that the cosine of $2\theta_{lp}$ is equal to:

$$\cos 2\theta_{lp} = U_{lp}^2 - V_{lp}^2 = \frac{\varepsilon_{lp}}{E_{lp}} \rightarrow 1 \quad (7.30)$$

Using this parameterization, the gap equation for the generalized form of the NJL model can be constructed. The gap equation in the generalized NJL model is similar to the gap equation in the original NJL model. The only difference is that now it is generalized to the multiple lepton case. The generalized gap equation is shown in Eq 7.31.

$$\Delta_l = \frac{4g_l}{V} \sum_{l'p'} \frac{g_{l'} \Delta_{l'}}{\sqrt{\tilde{m}_{l'}^2 + p^2}} = \frac{2g_l}{\pi^2} \sum_{l'} (g_{l'} \Delta_{l'} I_{l'}) \quad (7.31)$$

The integral I_l (given in Eq 7.32) in Eq 7.31 is also virtually identical to the integral I in the non-generalized case.

$$I_l = \int_0^\infty dp \frac{p^2}{\sqrt{p^2 + \tilde{m}_l^2}} \quad (7.32)$$

Its solution is:

$$I_l = \frac{1}{4} \left[\tilde{m}_l^2 \left(\ln \frac{\tilde{m}_l^2}{\mu_s^2} - 1 \right) + C_l \right] \quad (7.33)$$

So Eq 7.31 becomes:

$$\sum_{l'} g_{l'} \Delta_{l'} \left[\tilde{m}_{l'}^2 \left(\ln \frac{\tilde{m}_{l'}^2}{\mu_s^2} - 1 \right) + C_{l'} \right]_{VPT} = \frac{2\pi^2 \Delta_l}{g_l} \quad (7.34)$$

The kinetic energy of the system is given as:

$$4 \sum_{\vec{p}} \varepsilon_{lp} V_{lp}^2 = \frac{V}{\pi^2} J_l + \frac{\tilde{m}_l^2 V}{\mu_s^2} I_l \quad (7.35)$$

where the integral J_l is also divergent. Using the same regularization renormalization method used on I_l and J_l (from the non-generalized NJL model), the solution to J_l is found to be:

$$J_l = \frac{-1}{16} \left[\tilde{m}_l^4 \left(\ln \frac{\tilde{m}_l}{\mu_s^2} - \frac{3}{2} \right) + 2C'_l \tilde{m}_l^2 + D_l \right] \quad (7.36)$$

The regularization process gives the same conditions as before, with:

$$\frac{\partial E_{vac}}{\partial \Delta_l^2}|_{VPT} = \frac{V}{8\pi^2} (C_l - C'_l) = 0 \quad (7.37)$$

C_l in Eq 7.37 can be found from the gap equation, and substituted into Eq 7.37 to get:

$$\frac{\partial E_{vac}}{\partial \Delta_l^2}|_{VPT} = -\frac{V}{8\pi^2} \left[\tilde{m}_l^2 \left(\ln \frac{\tilde{m}_l^2}{\mu_s^2} - 1 \right) - \frac{2\pi^2}{G_l} + C'_l \right] = 0 \quad (7.38)$$

which leads to the result:

$$C_l = C'_l. \quad (7.39)$$

From these results, a complete formulation of the vacuum energy can be created. The vacuum energy formulation is:

$$E_{vac} = -\frac{V}{16\pi^2} \sum_l \left\{ \tilde{m}_l^4 \left(\ln \frac{\tilde{m}_l^2}{\mu_s^2} - \frac{3}{2} \right) + 2C'_l \tilde{m}_l^2 + D_l - \Delta_l^2 \left[\tilde{m}_l^2 \left(\ln \frac{\tilde{m}_l^2}{\mu_s^2} - 1 \right) + C_l \right] \right\} \\ - \frac{V}{16\pi^4} \left\{ \sum_{\nu} g_{\nu} \Delta_{\nu} \left[\tilde{m}_{\nu}^2 \left(\ln \frac{\tilde{m}_{\nu}^2}{\mu_s^2} - 1 \right) + C_{\nu} \right] \right\}^2 \quad (7.40)$$

with the condition:

$$E_{vac}|_{\Delta_l=0} = 0 \rightarrow D_l = -\delta_l^4 \left(\ln \frac{\delta_l^2}{\mu_s^2} - \frac{3}{2} \right) - 2\delta_l^2 C_l \quad (7.41)$$

being a result of the regularization process.

A new constant ξ is introduced at this stage of the theory. ξ describes the mass ratio of:

$$\xi = \frac{R}{r} \quad (7.42)$$

In order to further simplify the model, a variable T is proposed. T represents the sum of the mass ratios squared (for three flavors $f = 3$) of each lepton compared to the datum mass, or:

$$T = 1 + r^2 + R^2 = 1 + r^2 + \xi^2 r^2 \quad (7.43)$$

The sum of the natural logarithms of each mass ratio multiplied by each mass ratio squared is also formulated as a variable J :

$$J = r^2 \ln r + R^2 \ln R = r^2 \ln r + r^2 \xi^2 \ln r + r^2 \xi^2 \ln \xi \quad (7.44)$$

The variable U given in 7.45 is also necessary to consider, where the values of the arbitrary constants u , v , and w are waiting to be fixed later in the theory $\left(u = \frac{C_1}{m_1^2}, v = \frac{C_2}{m_1^2}, w = \frac{C_3}{m_1^2}\right)$.

$$U = u + rv + Rw \quad (7.45)$$

After introducing the seed mass δ_l and the corresponding variable $s = \frac{\delta_l^2}{m_1^2}$, the vacuum energy density \tilde{W} (shown in Eq 7.46) during the vacuum phase transition is $\frac{16\pi^2 E_{vac}}{Vm_1^4}$ and can be expressed in terms of U , T , J , and the variable y which is does not yet have a fixed value (see Eq 7.48).

$$\begin{aligned} \frac{16\pi^2 E_{vac}}{Vm_1^4} = \tilde{W}(x, s) = (s^2 - x^2) \left[\left(y - \frac{3}{2} \right) T + J \right] + [s^2 \ln s - x^2 \ln x] T + 2(s - x)U \\ - 4(s - x)x \left[(\ln x + y - 1)T + J + \frac{U}{x} \right] + \frac{f(s)}{a_1} (s - x)x^2 \left[(\ln x + y - 1)T + J + \frac{U}{x} \right]^2 \end{aligned} \quad (7.46)$$

Hence, the seed mass growing function $f(s)$ is inserted so that during the progression of the VPT, we take care of δ_1^2 being not m_1^2 as that in the expression $\frac{1}{a_1} = \frac{G_1 m_1^2}{\pi^2}$. $f(s)$ is chosen as the exponential function given in Eq 7.47.

$$f(s) = e^{\frac{-1}{s}+1} \quad (7.47)$$

The variable y in \tilde{W} is also waiting to be fixed to (see Appendix for more details):

$$y = \ln \frac{m_1^2}{\mu_s^2} \quad (7.48)$$

The vacuum energy density at the intermediate stage of the VPT (i.e. $|0_{NJL})$) can be described as $W(x)$ with $s \rightarrow 0$ ($f(s)$ is eliminated as well). So, using Eq 7.46 and Eq 7.47, the vacuum energy density at the end of the VPT ($W(x)$) is described in Eq 7.49.

$$\begin{aligned} W(x) = -x^2 \left[\left(y - \frac{3}{2} \right) T + J \right] - x^2 \ln(x)T - 2xU + 4x^2 \left[(\ln x + y - 1) T + J + \frac{U}{x} \right] \\ - \frac{x^3}{a_1} \left[(\ln x + y - 1) T + J + \frac{U}{x} \right]^2 \end{aligned} \quad (7.49)$$

Unfortunately, the equations derived from this which describe the mass ratios do not have a solution for r and ξ . This implies that, at least for the case when we consider just the electron, muon, and tau, this theory cannot propose a solution. However, this does not inhibit the proposal of additional leptons in order to satisfy the revised NJL theory.

7.1 Six Flavor Lepton Case

After examining the four and five lepton scenarios, it was found that they may not give a valid solution. So, a six lepton scenario was proposed by Ni and his colleagues (including this author). This six lepton scenario uses the ratios:

$$r = \frac{m_2}{m_1}, R = \frac{m_3}{m_1}, F = \frac{m_4}{m_1}, P = \frac{m_5}{m_1}, H = \frac{m_6}{m_1} \quad (7.50)$$

where m_2, m_3, m_4, m_5 , and m_6 are the masses of the final leptons created by the VPT, and m_1 is again the mass of the lepton chosen as the reference lepton. The values of T and J are then revised to become:

$$T = 1 + r^2 + R^2 + F^2 + P^2 + H^2 = 1 + r^2\{1 + \xi^2[1 + \kappa^2(1 + \rho^2)]\} \quad (7.51)$$

$$J = 1 + r^2 \ln r + R^2 \ln R + F^2 \ln F + P^2 \ln P + H^2 \ln H = 1 + r^2 \ln r \{1 + \xi^2 \ln \xi [1 + \kappa^2 \ln \kappa (1 + \rho^2 \ln \rho)]\} \quad (7.52)$$

where the ratios ξ, κ, ρ , and σ are:

$$\xi = \frac{R}{r}, \kappa = \frac{F}{R}, \rho = \frac{P}{F}, \sigma = \frac{H}{P} \quad (7.53)$$

Five final equations can then be formulated in terms of r, ξ, κ, ρ , and σ . Ni and his collaborators (including this author) are currently in the process of formulating this six lepton theory. This author was highly involved in the calculations of the four, five, and six lepton scenarios.

8 Summary and Discussion

While also constructing a model for Leptons, Ni and his colleagues (including this author) have been creating a generalized model of the NJL model for quarks. Physicists already know that at least six quarks exist, the up, down, top, bottom, charm, and strange quarks. Thus, it is necessary to begin the consideration of the quark model for six masses by considering five mass ratios with the datum mass (m_1):

$$r = \frac{m_2}{m_1}, R = \frac{m_3}{m_1}, F = \frac{m_4}{m_1}, P = \frac{m_5}{m_1}, H = \frac{m_6}{m_1} \quad (8.1)$$

where m_2, m_3, m_4, m_5 , and m_6 are the masses of five of the quark flavors. From this point, the preliminary theoretical framework is currently being constructed by Ni, his collaborators, and this author.

The implications of the generalized NJL model are very far reaching. One notable example is that of the 0^+ scalar bosons. Using the generalized NJL model, a theory can be constructed for these bosons which may be able to accurately predict their masses. Another one of the main implications may be in the use the quark vacuum has in discussing dark matter in the universe.

9 Appendix

The seed mass growing factor for leptons is given in Eq 7.47. The subsequent derivatives of this function multiplied by $(s - x)$ are given below, and assist in obtaining a solution from the energy density function. The first derivative is:

$$\left\{ \frac{\partial}{\partial s} [f(s)(s - x)] \right\}_{x=1, s=1} = 1 \quad (9.1)$$

The second derivative is:

$$\left\{ \frac{\partial^2}{\partial s^2} [f(s)(s - x)] \right\}_{x=1, s=1} = 2 \quad (9.2)$$

The third derivative is:

$$\left\{ \frac{\partial^3}{\partial s^3} [f(s)(s - x)] \right\}_{x=1, s=1} = -3 \quad (9.3)$$

The fourth derivative is:

$$\left\{ \frac{\partial^4}{\partial s^4} [f(s)(s - x)] \right\}_{x=1, s=1} = 4 \quad (9.4)$$

Using these results, the derivatives of the vacuum energy density (Eq 7.46) can be calculated. The results of these calculations can be used to determine some of the unknown constants in the expression for the vacuum energy density. The results are given below as:

$$\tilde{W}_{s11} = 0 \rightarrow [(y_1 - 1)T + J + U]_{x=1} = 2a_1, a_1 = \frac{\pi^2}{G_1 m_1^2} \quad (9.5)$$

$$\tilde{W}_{ss11} = 0 \rightarrow y \rightarrow y_1 = -\frac{J}{T} - \frac{2}{3} \quad (9.6)$$

$$\tilde{W}_{sss11} = 0 \rightarrow U = 2T \rightarrow a_1 = \frac{T}{6} \quad (9.7)$$

$$\tilde{W}_{ssss11} = \frac{2}{3}T > 0 \quad (9.8)$$

For the quark case, four separate seed mass renormalization functions must be considered. These four (the column labels of the table) and their derivatives (the row labels of the table) are given in Table 9.1.

	$f^{3/2}(s)(s-x)$	$f^{3/2}(s)(s^2-x^2)$	$f^{3/2}(s)s^2 \ln s$	$f^{5/2}(s)(s-x)$
$\frac{\partial}{\partial s}(\dots)_{11}$	1	2	1	1
$\frac{\partial^2}{\partial s^2}(\dots)_{11}$	3	8	6	5
$\frac{\partial^3}{\partial s^3}(\dots)_{11}$	$-\frac{9}{4}$	$\frac{9}{2}$	$\frac{53}{4}$	$\frac{15}{4}$
$\frac{\partial^4}{\partial s^4}(\dots)_{11}$	$-\frac{9}{2}$	-18	-8	$-\frac{55}{2}$

Table 9.1: The Seed Mass Growing Factor for Quarks

The results for the quark case have yet to be utilized in the actual theoretical formulation of the generalized NJL model for quarks, but, they are anticipated to be very important components of the theory.

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